Helly-type Theorems for Hollow Axis-aligned Boxes

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Abstract. A *hollow axis-aligned box* is the boundary of the cartesian product of d compact intervals in $\mathbb{R}^d$. We show that for $d \geq 3$, if any 2^d of a collection of hollow axis-aligned boxes have non-empty intersection, then the whole collection has non-empty intersection; and if any 5 of a collection of hollow axis-aligned rectangles in $\mathbb{R}^2$ have non-empty intersection, then the whole collection has non-empty intersection. The values 2^d for $d \geq 3$ and 5 for $d = 2$ are the best possible in general. We also characterize the collections of hollow boxes which would be counterexamples if 2^d were lowered to $2^d - 1$, and 5 to 4, respectively.

1. General Notation and Definitions

We denote the cardinality of a set S by $\#S$. Let $\Pi(\mathbf{S}, k)$ denote the property that any subcollection of $\mathbf{S}$ of at most k sets has non-empty intersection (where k is any positive integer), and $\Pi(\mathbf{S})$ the property that $\mathbf{S}$ has non-empty intersection. For any set $S \subseteq \mathbb{R}^d$, we denote the convex hull, interior and boundary by $\operatorname{co} S, \operatorname{int} S$ and $\operatorname{bd} S$, respectively. An *axis-aligned box* in $\mathbb{R}^d$ is the cartesian product of d compact intervals, i.e. a set of the form

$$\prod_{i=1}^{d}[a_i, b_i] = \big\{(x_1, \dots, x_d) \in \mathbb{R}^d : a_i \leq x_i \leq b_i, i = 1, \dots, d\big\}, \quad (a_i < b_i).$$

An *axis-aligned hollow box* in $\mathbb{R}^d$ is the boundary of a box, i.e. a set of the form

$$\operatorname{bd} \prod_{i=1}^{d}[a_i, b_i], \quad (a_i < b_i).$$

In the rest of the paper, the word *axis-aligned* is implicit whenever we refer to boxes or hollow boxes. In the next section we state our results (Theorems 1 and 2), together with examples showing that they are the best possible. In Section 3 we derive a combinatorial lemma needed in the proofs of these theorems in Section 4.

2. Helly-type Theorems

A Helly-type theorem may be loosely described as an analogue of

Helly's theorem ([**6**]). *Let* $\mathbf{S}$ *be a collection of convex sets in* $\mathbb{R}^d$ *that is finite, or contains at least one compact set. Then*

$$\Pi(\mathbf{S}, d+1) \implies \Pi(\mathbf{S}).$$

□

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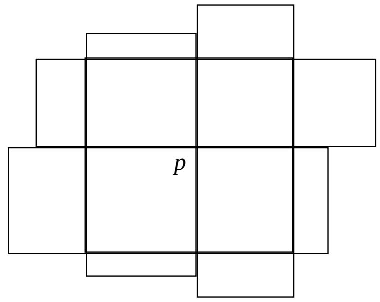

FIGURE 1. Five rectangles with no common boundary point, yet any 4 have a common boundary point

There is an abundance of literature on Helly-type theorems; see the surveys [**1, 3, 5**]. Most of these analogues consider collections of *convex* sets, exactly as in Helly's Theorem. Here are two examples where non-convex sets are considered.

THEOREM (Motzkin [**8, 2**]). *Let* $\mathbf{S}$ *be a collection of sets in* $\mathbb{R}^d$*, each of which is the set of common zeroes of a set of real polynomials in* d *variables of degree at most* k*. Then*

$$\Pi(\mathbf{S}, \tbinom{d+k}{k}) \implies \Pi(\mathbf{S}).$$

$\square$

THEOREM (Maehara [**7, 4**]). *Let* $\mathbf{S}$ *be a collection of at least* $d+3$ *euclidean spheres in* $\mathbb{R}^d$*. Then*

$$\Pi(\mathbf{S}, d+1) \implies \Pi(\mathbf{S}).$$

$\square$

In both these theorems the sets are algebraic. In this paper we find Helly-type theorems for certain non-algebraic sets, namely hollow boxes. It is well-known (and immediately follows from the one-dimensional Helly theorem) that for any collection $\mathbf{S}$ of boxes in $\mathbb{R}^d$,

$$\Pi(\mathbf{S}, 2) \implies \Pi(\mathbf{S}).$$

If we want the boxes to intersect only in their boundaries, then the value 2 has to be greatly enlarged, as the following examples show.

EXAMPLE 1. *A class of collections* $\mathbf{S}$ *of hollow boxes in* $\mathbb{R}^d$ *such that* $\Pi(\mathbf{S}, 2d)$ *holds, but not* $\Pi(\mathbf{S}, 2d+1)$.

Choose any box $B = \prod_{i=1}^d [x_i^0, x_i^1]$, (where $x_i^0 < x_i^1$), and $p = (p_1, \dots, p_d) \in \operatorname{int} B$. For $i = 1, \dots, d$ and $j = 0, 1$, let F_i^j denote the facet of B contained in the hyperplane $\{x \in \mathbb{R}^d : x_i = x_i^j\}$. Let $\mathbf{S}$ be any collection of hollow boxes such that

(1) $\operatorname{bd} B \in \mathbf{S}$,

(2) $p \in D$ for all $D \in \mathbf{S} \setminus \{\operatorname{bd} B\}$,

(3) for each $D \in \mathbf{S} \setminus \{\operatorname{bd} B\}$ there is a facet of B contained in D,

(4) for each facet F of B there exists some $D \in \mathbf{S} \setminus \{\operatorname{bd} B\}$ such that $F \subseteq D$.

It is clear that there exist such collections $\mathbf{S}$, (even infinite ones provided $d \neq 1$). Note that the facet in (3) is unique, by (2). See Figure 1 for an example in $\mathbb{R}^2$.

Choose any subcollection $\mathbf{T} \subseteq \mathbf{S}$ of $2d$ hollow boxes. If $\operatorname{bd} B \notin \mathbf{T}$, then by (2), $\bigcap_{D \in \mathbf{T}} D \neq \emptyset$. Otherwise, by (3), there is a facet of B not contained in any $D \in \mathbf{T} \setminus \{\operatorname{bd} B\}$, say F_1^0. Then it easily follows from (2) and (3) that $(x_1^1, p_2, p_3, \dots, p_d) \in \bigcap_{D \in \mathbf{T}} D$. It follows that $\Pi(\mathbf{S}, 2d)$ holds.

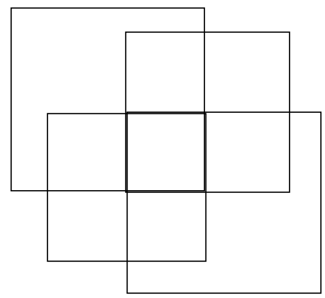

FIGURE 2. Four rectangles with no common boundary point, yet any 3 have a common boundary point

Secondly, use (4) to choose for each facet F_i^j of B a $D_i^j \in \mathbf{S}$ containing F_i^j. Then $F_i^{1-j} \cap D_i^j = \emptyset$ by (2). It follows that $(\operatorname{bd} B) \cap \bigcap_{i=1}^d (D_i^0 \cap D_i^1) = \emptyset$, and $\Pi(\mathbf{S}, 2d+1)$ does not hold. $\square$

EXAMPLE 2. *A class of collections* $\mathbf{S}$ *of hollow boxes in* $\mathbb{R}^d$ *such that* $\Pi(\mathbf{S}, 2^d - 1)$ *holds, but not* $\Pi(\mathbf{S}, 2^d)$.

Let $B = \prod_{i=1}^d [x_i^0, x_i^1], (x_i^0 < x_i^1)$, and let $\mathbf{S}$ be any collection of hollow boxes such that

(5) $B \subseteq \operatorname{co} D$ for all $D \in \mathbf{S}$,

(6) for each vertex v of B there exists a $D \in \mathbf{S}$ not containing v,

(7) each $D \in \mathbf{S}$ contains all the vertices of B except at most one.

It is clear thus there exist such collections, even infinite ones. See Figure 2 for an example in $\mathbb{R}^2$. Given a subcollection of $2^d - 1$ hollow boxes, then by (7), some vertex of B is contained in all these boxes. Thus $\Pi(\mathbf{S}, 2^d - 1)$ holds.

Secondly, (6) gives a subcollection of 2^d boxes D_v with $v \notin D_v$. But then, using also (5), it follows from Lemma 4.2 that for any vertex w of B, $\bigcap_{v \neq w} D_v = \{w\}$. Thus, $\bigcap_v D_v = \emptyset$, and $\Pi(\mathbf{S}, 2^d)$ does not hold. $\square$

The following two theorems show that the collections in Example 1 in the case $d = 2$, and the collections in Example 2 in the case $d \geq 3$ are the worst cases.

THEOREM 1. *Let* $\mathbf{S}$ *be a collection of hollow boxes in* $\mathbb{R}^2$. *Then*

$$\Pi(\mathbf{S}, 5) \implies \Pi(\mathbf{S}).$$

If $\mathbf{S}$ *is furthermore not of the form in Example 1, then*

$$\Pi(\mathbf{S}, 4) \implies \Pi(\mathbf{S}).$$

THEOREM 2. *Let* $d \geq 3$, *and* $\mathbf{S}$ *a collection of hollow boxes in* $\mathbb{R}^d$. *Then*

$$\Pi(\mathbf{S}, 2^d) \implies \Pi(\mathbf{S}).$$

If $\mathbf{S}$ *is furthermore not of the form in Example 2, then*

$$\Pi(\mathbf{S}, 2^d - 1) \implies \Pi(\mathbf{S}).$$

Note that in $\mathbb{R}^1$, a hollow box is a two-point set. It is trivially seen that for a collection $\mathbf{S}$ of two-point sets,

$$\Pi(\mathbf{S}, 2) \implies \Pi(\mathbf{S}),$$

except if $\mathbf{S} = \{\{a, b\}, \{b, c\}, \{c, a\}\}$ for some distinct elements a, b, c, i.e. if $\mathbf{S}$ is as in Example 1.

3. Combinatorial Preparation

A *string of length* d *over the alphabet* A is any d-tuple from A^d, and is written as $\varepsilon = \varepsilon_1 \dots \varepsilon_d$. We say that ε_i is in *position* i. A *pattern* is a string over $\{0, 1, *\}$. A string $\varepsilon_1 \dots \varepsilon_d$ over $\{0, 1\}$ *matches* a pattern $\rho_1 \dots \rho_d$ if for all $i = 1, \dots, d$, $\rho_i = 0 \Rightarrow \varepsilon_i = 0$ and $\rho_i = 1 \Rightarrow \varepsilon_i = 1$. Thus, a $*$ in a pattern is a "wildcard" matching 0 or 1. A *cover* of $\{0, 1\}^d$ is a set of patterns $\mathbf{C} \subseteq \{0, 1, *\}^d$ such that any string in $\{0, 1\}^d$ matches some pattern in $\mathbf{C}$. A *minimal cover* of $\{0, 1\}^d$ is a cover $\mathbf{C}$ of $\{0, 1\}^d$ such that no proper subset of $\mathbf{C}$ is a cover of $\{0, 1\}^d$.

Lemma 1. *Let* $\mathbf{C}$ *be a minimal cover of* $\{0, 1\}^d$. *Then, for each* $i = 1, \dots, d$, $E_i := \{\varepsilon_i : \varepsilon_1 \dots \varepsilon_d \in C\}$ *is equal to either* $\{*\}$, $\{0, 1\}$ *or* $\{0, 1, *\}$. *Let* $s := \#\{i : E_i = \{*\}\}$. *Then* $\#\mathbf{C} \leq 2^{d-s}$, *with equality iff* $\mathbf{C} = \{\varepsilon : \varepsilon_i = * \text{ for all } i \in J\}$ *for some* $J \subseteq \{1, 2, \dots, d\}$ *with* $\#J = s$.

Proof. We first show that any minimal cover $\mathbf{C}$ satisfies $\#\mathbf{C} \leq 2^d$, with equality iff $\mathbf{C} = \{0, 1\}^d$. For each pattern $\rho \in \mathbf{C}$, the set $\mathbf{C} \setminus \{\rho\}$ is not a cover of $\{0, 1\}^d$, and there exists a string $\varepsilon_\rho \in \{0, 1\}^d$ that matches ρ but does not match any other pattern in $\mathbf{C}$. Thus,

$$\phi : \mathbf{C} \to \{0, 1\}^d; \rho \mapsto \varepsilon_\rho$$

is an injection, and $\#\mathbf{C} \leq 2^d$. If equality holds, ϕ is a bijection, and any string in $\{0, 1\}^d$ matches a unique pattern in $\mathbf{C}$. Thus $\mathbf{C}$ defines a partition of $\{0, 1\}^d$: a block of the partition consists of all strings matching a given pattern in $\mathbf{C}$. Since there are 2^d blocks, each block must contain exactly 1 element. Thus no pattern in $\mathbf{C}$ contains a $*$, and $\mathbf{C} = \{0, 1\}^d$.

Secondly, we show that if 0 does not occur in the first position of any string in $\mathbf{C}$, there are only $*$s in the first position. Let

$$\mathbf{C}^* = \{\varepsilon_2 \dots \varepsilon_n : *\varepsilon_2 \dots \varepsilon_n \in \mathbf{C}\}.$$

It is easily seen that $\mathbf{C}^*$ is a cover for $\{0, 1\}^{d-1}$: For any $\varepsilon \in \{0, 1\}^{d-1}$, 0ε matches some pattern in $\mathbf{C}$ starting with $*$. But then, by putting back $*$ in the first position of every pattern in $\mathbf{C}^*$, we already obtain a cover of $\{0, 1\}^d$. Thus, 1 does not occur in the first position in any string in $\mathbf{C}$. Similarly, if 1 does not occur in the first position, then there are again only $*$s in the first position.

Finally, to complete the proof, delete the positions for which $E_i = \{*\}$, to obtain $\mathbf{C}' \subseteq \{0, 1, *\}^{d-s}$. Then $\mathbf{C}'$ is clearly a minimal cover of $\{0, 1\}^{d-s}$, and $\#\mathbf{C} = \#\mathbf{C}'$. Now apply the first part of the proof. □

We omit the proof of the following elementary inequality.

Lemma 2. *Let* $d \geq s \geq 0$ *be integers. Then* $2^{d-s} < 2^d - 2s$, *except in the following cases:*

1. *If* $(d, s) = (1, 1)$ *or* $(d, s) = (2, 2)$, *the opposite inequality holds;*
2. *If* $s = 0$, *or* $(d, s) = (2, 1)$, *there is equality.* □

Lemma 3. *With the hypothesis of Lemma 1,* $\#\mathbf{C} < 2^d - 2s$, *except in the following cases:*

1. *If* $\mathbf{C} = \{*\}$ *or* $\mathbf{C} = \{**\}$ *then* $\#\mathbf{C} > 2^d - 2s = 0$;
2. *If* $\mathbf{C} = \{0, 1\}^d$ *or* $\mathbf{C} = \{0*, 1*\}$ *or* $\mathbf{C} = \{*0, *1\}$ *then* $\#\mathbf{C} = 2^d - 2s$.

Proof. It is easy to check everything for $d = 1$ and $d = 2$: The only minimal covers for $d = 1$ are $\{*\}$ and $\{0, 1\}$, and for $d = 2$, are equivalent (up to permutation of the positions, and interchange of 0 and 1) to one of

$$\{**\}, \{0*, 1*\}, \{0*, 10, 11\}, \{0*, *0, 11\}, \{00, 01, 10, 11\}.$$

For $d \geq 3$, if $s \geq 1$, then $\#\mathbf{C} \leq 2^{d-s} < 2^d - 2s$, by Lemmas 1 and 2. Otherwise, $s = 0$, and by Lemma 1, $\#\mathbf{C} < 2^d$ unless $\mathbf{C} = \{0,1\}^d$. □

4. Proofs of Theorems 1 and 2

We first prove a rather technical lemma, which gives some insight into the (not easily visualizable) intersections of hollow boxes.

LEMMA 4. *Let $B = \prod_{i=1}^d [x_i^0, x_i^1]$, with $x_i^0 \leq x_i^1$ for each $i = 1, \dots, d$. (Thus B is not necessarily full-dimensional.) For each string $\varepsilon \in \{0,1\}^d$, let $x_\varepsilon := (x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \dots, x_d^{\varepsilon_d})$, and let D_ε be a hollow box such that $x_\varepsilon \notin D_\varepsilon$ and $B \subseteq \operatorname{co} D_\varepsilon$. (Thus $\{x_\varepsilon : \varepsilon \in \{0,1\}^d\}$ is the vertex set of B, with repetitions if $\dim B < d$.) Then,*

1. *$B \cap \bigcap_\varepsilon D_\varepsilon = \emptyset$,*
2. *for any $\gamma \in \{0,1\}^d$, $B \cap \bigcap_{\varepsilon \neq \gamma} D_\varepsilon \subseteq \{x_\gamma\}$,*
3. *for any $\gamma, \delta \in \{0,1\}^d$,*

$$B \cap \bigcap_{\varepsilon \neq \gamma,\delta} D_\varepsilon \subseteq \begin{cases} \operatorname{co}\{x_\gamma, x_\delta\} & \textit{if } x_\gamma \textit{ and } x_\delta \textit{ differ in exactly one coordinate,} \\ \{x_\gamma, x_\delta\} & \textit{otherwise.} \end{cases}$$

PROOF. Clearly, part 1 follows from part 2: If B is a single point, each D_ε is disjoint from B. Otherwise, choose γ, γ' such that $x_\gamma \neq x_{\gamma'}$. Then, by part 2, $B \cap \bigcap_\varepsilon D_\varepsilon = \emptyset$.

Although part 2 also easily follows from part 3, we first prove part 2, as it clears the way for a proof of part 3. For each ε, write $D_\varepsilon = \operatorname{bd} \prod_{i=1}^d [a_i^\varepsilon, b_i^\varepsilon]$. Let $x = (x_1, x_2, \dots, x_d) \in B \cap \bigcap_{\varepsilon \neq \gamma} D_\varepsilon$. Then $x_i^0 \leq x_i \leq x_i^1$ for each i. Define ε by

$$\varepsilon_i := \begin{cases} \gamma_i & \text{if } x_i = x_i^{\gamma_i}, \\ 1 - \gamma_i & \text{otherwise.} \end{cases}$$

Since $x_\varepsilon \subseteq B \subseteq \operatorname{co} D_\varepsilon$, but $x_\varepsilon \notin D_\varepsilon$, we have $a_i^\varepsilon \leq x_i^0 \leq x_i^1 \leq b_i^\varepsilon$ and $a_i^\varepsilon < x_i^{\varepsilon_i} < b_i^\varepsilon$ for all i. If $\varepsilon_i = \gamma_i$, then $x_i^{\varepsilon_i} = x_i^{\gamma_i} = x_i$. If $\varepsilon_i = 1 - \gamma_i$, then $x_i \neq x_i^{\gamma_i}$, and either $\gamma_i = 1$ and $x_i^{\varepsilon_i} = x_i^0 \leq x_i < x_i^1$, or $\gamma_i = 0$ and $x_i^{\varepsilon_i} = x_i^1 \geq x_i > x_i^0$. In all cases, $a_i^\varepsilon < x_i < b_i^\varepsilon$, and it follows that $x \notin D_\varepsilon$. Thus $\varepsilon = \gamma$, and $x_i = x_i^{\gamma_i}$ for all i. It follows that $x = x_\gamma$.

Now let $x \in B \cap \bigcap_{\varepsilon \neq \gamma,\delta} D_\varepsilon$, and suppose $x \neq x_\gamma, x_\delta$. Let j be any position such that $x_j \neq x_j^{\gamma_j}$. Define ε by

$$\varepsilon_i := \begin{cases} 1 - \gamma_i & \text{if } i = j, \\ \delta_i & \text{if } x_i = x_i^{\delta_i}, i \neq j, \\ 1 - \delta_i & \text{if } x_i \neq x_i^{\delta_i}, i \neq j. \end{cases}$$

As in the proof of part 2, for each i we obtain $a_i^\varepsilon < x_i < b_i^\varepsilon$, and therefore, $x \notin D_\varepsilon$. Thus, $\varepsilon = \gamma$ or $\varepsilon = \delta$. But, since $\varepsilon_j \neq \gamma_j$, we must have $\varepsilon = \delta$. Thus, $\gamma_j = 1 - \delta_j$, and for all $i \neq j$, $x_i = x_i^{\delta_i}$. Since $x \neq x_\delta$ we then must have $x_j \neq x_j^{\delta_j}$. By repeating the above argument with x_δ instead of x_γ, we also obtain that for all $i \neq j$, $x_i = x_i^{\gamma_i}$. It follows that $x \in \operatorname{co}\{x_\gamma, x_\delta\}$, and x_γ and x_δ differ in only one coordinate. □

PROOF OF THEOREM 2. Note that the first part of the theorem follows from the second part, since $\Pi(\mathbf{S}, 2^d)$ does not hold in Example 2. By compactness, we only have to prove the theorem for finite $\mathbf{S}$. We assume that $\Pi(\mathbf{S}, 2^d - 1)$. Let $B = \bigcap_{D \in \mathbf{S}} \operatorname{co} D = \prod_{i=1}^d [x_i^0, x_i^1]$. (Since any two Ds intersect, $x_i^0 \leq x_i^1$ for all i.) We denote the vertices of B by x_ε, $\varepsilon \in \{0,1\}^d$, as in Lemma 4. We now show that if $x_\varepsilon \notin \bigcap_{D \in \mathbf{S}} D$ for all ε, then $\mathbf{S}$ is as in Example 2.

For each ε, choose $D_\varepsilon = \operatorname{bd}\prod_{i=1}^d [a_i^\varepsilon, b_i^\varepsilon] \in \mathbf{S}$ such that $x_\varepsilon \notin D_\varepsilon$, and let

$$X_\varepsilon := \{x_\delta : \delta \in \{0,1\}^d, x_\delta \notin D_\varepsilon\}.$$

Then $X_\varepsilon = \{x_\delta : \delta \text{ matches } \rho_\varepsilon\}$, where $\rho_\varepsilon = \rho_1 \dots \rho_d$ is the pattern defined by

$$\rho_i := \begin{cases} 0 & \text{if } a_i^\varepsilon < x_i^0 \text{ and } x_i^1 = b_i^\varepsilon, \\ 1 & \text{if } a_i^\varepsilon = x_i^0 \text{ and } x_i^1 < b_i^\varepsilon, \\ * & \text{if } a_i^\varepsilon < x_i^0 \text{ and } x_i^1 < b_i^\varepsilon. \end{cases}$$

Thus $\mathbf{C} := \{\rho_\varepsilon : \varepsilon \in \{0,1\}^d\}$ is a cover of $\{0,1\}^d$. If $\rho_\varepsilon = \rho_{\varepsilon'}$, then $x_{\varepsilon'} \notin D_\varepsilon$, so we may choose the D_εs such that if $\rho_\varepsilon = \rho_{\varepsilon'}$, then $D_\varepsilon = D_{\varepsilon'}$. We now write D_ρ for D_ε whenever $\rho = \rho_\varepsilon \in \mathbf{C}$. Let $\mathbf{C}'$ be a minimal cover contained in $\mathbf{C}$. For each $\varepsilon \in \{0,1\}^d$ there now exists a $\rho \in \mathbf{C}'$ matching ε such that $x_\varepsilon \notin D_\rho$. Applying Lemma 4.1 to $\{D_\rho : \rho \in \mathbf{C}'\}$, we find $B \cap \bigcap_\rho D_\rho = \emptyset$. Let $J \subseteq \{1, \dots, d\}$ be the set of positions in which there are only $*$s in $\mathbf{C}'$. For each $j \in J$, choose $D_j^0 = \operatorname{bd}\prod_{i=1}^d [r_i^j, s_i^j]$ and $D_j^1 = \operatorname{bd}\prod_{i=1}^d [t_i^j, u_i^j]$ from $\mathbf{S}$ such that $r_j^j = x_j^0$ and $u_j^j = x_j^1$ (which is possible since $\mathbf{S}$ is finite). Since (by Lemma 1) for each $i \notin J$ there exist $\rho, \rho' \in \mathbf{C}'$ such that $\rho_i = 0$ and $\rho_i' = 1$, we obtain

$$\bigcap_{j \in J} (\operatorname{co} D_j^0 \cap \operatorname{co} D_j^1) \cap \bigcap_{\rho \in \mathbf{C}'} \operatorname{co} D_\rho = B.$$

Thus, letting $\mathbf{T} := \{D_\rho : \rho \in \mathbf{C}'\} \cup \{D_j^0, D_j^1 : j \in J\}$, we obtain $\bigcap_{D \in \mathbf{T}} D = \emptyset$. Thus, $\#\mathbf{T} \geq 2^d$. Also, $\#\mathbf{T} \leq \#\mathbf{C}' + 2\#J$. Thus, by Lemma 3, $\mathbf{C}' = \{0,1\}^d$. It follows that $x_\delta \notin D_\varepsilon$ iff $\delta = \varepsilon$. Thus, all x_εs are distinct, and B is full-dimensional. Also, $J = \emptyset$ and $B = \bigcap_\varepsilon \operatorname{co} D_\varepsilon$. In fact, if we take any ε and ε' which differ in each position, then $B = \operatorname{co} D_\varepsilon \cap \operatorname{co} D_{\varepsilon'}$.

We already have that $\mathbf{S}$ satisfies (5) and (6) in Example 2. Consider any $D \in \mathbf{S}$ with $D \neq D_\varepsilon$ for all ε. Suppose there exist distinct γ, δ such that $x_\gamma, x_\delta \notin D$. By Lemma 4.3, $D \cap B \cap \bigcap_{\varepsilon \neq \gamma, \delta} D_\varepsilon = \emptyset$. But there exist $\varepsilon, \varepsilon' \notin \{\gamma, \delta\}$ differing in each position. Thus $\bigcap_{\varepsilon \neq \gamma,\delta} D_\varepsilon \subseteq B$, and $D \cap \bigcap_{\varepsilon \neq \gamma,\delta} D_\varepsilon = \emptyset$, contradicting $\Pi(\mathbf{S}, 2^d - 1)$. Thus D contains all x_εs, except at most one, and (7) is satisfied. □

Proof of Theorem 1. Proceeding as in the proof of Theorem 2, we assume that $\Pi(\mathbf{S}, 4)$ holds, and that no vertex of B is in $\bigcap_{D \in \mathbf{S}} D$, and obtain $\mathbf{C}' = \{**\}$ and $\#\mathbf{T} = 5$.

We now show that $\mathbf{S}$ is as in Example 1. Since $\mathbf{C}' = \{**\}$, there is only one D_ρ, say $D = D_{**}$, which is disjoint from B. Also, $\mathbf{T} = \{D_1^0, D_1^1, D_2^0, D_2^1, D\}$, with the D_j^is as in the proof of Theorem 2. Thus $\bigcap_{i,j} \operatorname{co} D_j^i = B$.

Suppose that for each $\varepsilon \in \{0,1\}^2$ there exists a D_j^i not containing x_ε. Then by Lemma 4.1, $\bigcap_{i,j} D_j^i = \emptyset$, contradicting $\Pi(\mathbf{S}, 4)$. Thus, some $x_\varepsilon \in \bigcap_{i,j} D_j^i$, say x_{00}.

Suppose that B is two-dimensional, i.e. $x_1^0 < x_1^1$ and $x_2^0 < x_2^1$. Then, since $x_{00} \in D_1^1$, D_1^1 contains at least two sides of B, and it follows that $B = \operatorname{co} D_1^1 \cap \operatorname{co} D_2^0 \cap \operatorname{co} D_2^1$ or $B = \operatorname{co} D_1^1 \cap \operatorname{co} D_1^0 \cap \operatorname{co} D_2^1$. Thus $D_1^1 \cap D_2^0 \cap D_2^1 \cap D = \emptyset$ or $D_1^1 \cap D_1^0 \cap D_2^1 \cap D = \emptyset$, both cases contradicting $\Pi(\mathbf{S}, 4)$.

Suppose B is one-dimensional, say $x_1^0 < x_1^1$ and $x_2^0 = x_2^1$. Then $D_2^0 \cap D_2^1$ is a horizontal segment containing B. If D_1^1 intersects $D_2^0 \cap D_2^1$ only in x_{00} and x_{10}, then $D_1^1 \cap D_2^0 \cap D_2^1 \cap D = \emptyset$, a contradiction. Thus, $B \subseteq D_1^1$. We may assume that D_2^1 and D_1^1 are on opposite sides of B (otherwise consider D_2^0 and D_1^1). Then $D_1^0 \cap D_1^1 \cap D_2^1 \subseteq B$ and $D_1^0 \cap D_1^1 \cap D_2^1 \cap D = \emptyset$, a contradiction.

Thus B is zero-dimensional, say $B = \{p\}$, where $p = x_{00} = (x_1, x_2)$ and $x_1 = x_1^0 = x_1^1$, $x_2 = x_2^0 = x_2^1$. Then $D_1^0 \cap D_1^1$ is a vertical line segment through p which must intersect $D_2^0 \cap D$ in a point $b \neq p$, and $D_2^1 \cap D$ in a point $a \neq p$. Similarly, $D_1^0 \cap D_1^1$ is a horizontal segment through p which must intersect $D_1^0 \cap D$

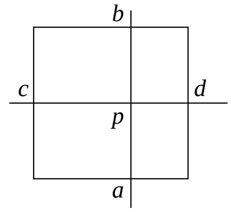

FIGURE 3.

in a point $d \neq p$, and $D_1^1 \cap D$ in a point $c \neq p$. See Figure 3. Now $\mathbf{S}$ already satisfies (1) and (4) of Example 1, if we take B there as $\operatorname{co} D$.

Consider any $E \in \mathbf{S} \setminus \mathbf{T}$. By considering the intersection of three sets at a time from $\mathbf{T}$, we see that E must intersect each of the sets $\{a, b\}$, $\{c, d\}$, $\{p, a\}$, $\{p, b\}$, $\{p, c\}$, $\{p, d\}$. If $p \notin E$, then $a, b, c, d \in E$, and $E = D$, a contradiction.

Thus $p \in E$, and (2) is satisfied. Also, $a \in E$ or $b \in E$. We may assume without loss that $a \in E$, and similarly, $c \in E$. But then, since $E \cap D \cap D_2^0 \cap D_1^0 \neq \emptyset$, we must have either $b \in E$ or $d \in E$, and (3) is satisfied. It follows that $\mathbf{S}$ is as in Example 1. □

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